Self-dual solutions of Yang-Mills theory on Euclidean AdS space

Özgür Sarıoğlu[*] and Bayram Tekin[†]

Department of Physics, Faculty of Arts and Sciences,
Middle East Technical University, 06531, Ankara, Turkey

(Dated: May 11, 2009)

We find non-trivial, time-dependent solutions of the (anti) self-dual Yang-Mills equations in the four dimensional Euclidean Anti-de Sitter space. In contrast to the Euclidean flat space, the action depends on the moduli parameters and the charge can take any non-integer value.

PACS numbers: 04.62.+v, 11.15.Kc, 11.15.Tk

I. INTRODUCTION

Finite action self-dual solutions with integer topological charge (instantons) of the Euclidean Yang-Mills (YM) theory in flat space ($\mathbb{R}^4$), and their tunneling interpretation between the classical minima (in fact zeros) of the potential is well established. [See [1] which compiles the original articles.] Once we depart from the Euclidean flat space, self-dual solutions are often drastically modified, if they are not totally wiped out. For example, on a four dimensional hypertorus T^4 [2], one has many different possibilities with non-integer Pontryagin number (topological charge) depending on the boundary conditions on the gauge fields. [For instantons on Taub-NUT space, see [3] and on $\mathbb{H}^3 \times \mathbb{R}$, see [4].]

There is, of course, a good motivation to depart from flat space and study self-dual YM theory in various curved backgrounds. For example, to define the finite temperature theory, one works on $S^1 \times \mathbb{R}^3$ [5]. The resulting self-dual solutions are called (untwisted) calorons, which come with integer topological charges and provide a tunneling interpretation [6] in the Weyl gauge. [It is not clear if the "twisted" calorons of [7, 8] allow for such an interpretation.] On a generic four dimensional Riemannian manifold, one can obtain some general statements [9], but one needs the explicit form of the solutions to actually utilize the self-dual solutions in physical problems beyond the semi-classical region.

Obviously the most relevant curved spaces are the ones that appear as solutions to General Relativity, with or without a cosmological constant. In principle, the effect of gravity on the perturbative sector of quantum field theories is expected to be quite weak, but this need not be so in the non-perturbative sector. Gravity usually brings in length scales and may also introduce new topologies other than that of the flat space, which in turn affects the non-perturbative solutions. This said, on a quantum mechanical system (not a field theoretical one), one does not expect gravity to have much effect on tunneling. For example, one can consider the case of the one dimensional double well potential $V(x) = (x^2 - 1)^2$ as a toy model of tunneling. Adding a constant gravitational potential $V_g(z) = -mgz$, turns it into a two dimensional tunneling problem. However the change is not dramatic: There will be new paths for tunneling. On the other hand, when one is interested in the vacuum of a field theory, such as YM theory, the effect of gravity becomes highly non-trivial. Arguably, the most relevant example is the YM theory in the Euclidean Schwarzschild background. It was shown in [10] that all the previously obtained solutions [11, 12] are static (i.e.

---

[*]Electronic address: sarioglu@metu.edu.tr
[†]Electronic address: btekin@metu.edu.tr

there is no dependence on the Euclidean time) which give rise to a constant potential. Thus they are solitons (monopoles and dyons) and are not instantons. [See [13] for more recent work.] It was conjectured in [10] that there are no YM instantons with a time dependent potential in the Euclidean Schwarzschild background. [See [14] for related work.]

In this paper, we shall present self-dual solutions to the $SU(2)$ YM theory in the four dimensional Euclidean Anti-de Sitter space[1]. [Euclidean de Sitter space can also be treated in the way we do here, however with two major modifications: In this space, time is compactified and one should stick to the region of the space inside the cosmological horizon.] In earlier works [16], *time-independent* solutions were constructed, but here we will present time-dependent solutions that do have a non-constant YM potential.

YM theory in four dimensions is conformal, thus the self-duality equations are intact under a conformal scaling of the metric. Hence a naive approach would yield that in AdS (which is conformal to the flat space) the usual instanton solutions are pretty much intact and no serious modifications are to be expected. However, this is not correct since the AdS space is in fact conformal to the unit ball, which means that the boundary is at a finite distance for timelike geodesics (of course, we really have "Euclidean" time here, but it is clear that the boundary effects will be quite important). Before we explicitly study how the boundary effects modify the topological charge of the solution, let us note that it has been known for a longtime that the finite action self-dual solutions are not necessarily classified by integer topological charge: In [17], it was shown that fractionally charged (specifically charge-3/2) instantons exist if one removes the condition on the continuity of the group-valued function g, for which YM connection on $\mathbb{R}^4$ asymptotically becomes $A \to g^{-1}dg$. Besides the continuity assumption, one also assumes that $g(\vec{x}) \to 1$ as $|\vec{x}| \to \infty$ leading to a compactification of $\mathbb{R}^3$ to S^3, and immediately making it transparent (for the usual instantons) that one has integer topological charge corresponding to the winding number of the maps $g : S^3 \to S^3$. As argued in [18], such a boundary condition is quite natural for the flat Euclidean space, but it need not be so in other spaces. AdS is one such example where the existence of the boundary at a finite distance completely modifies the instanton solutions [19, 20].

The outline of the paper is as follows: In section II, we set the stage for static, spherically symmetric Euclidean spaces and derive the self-duality equations for the $SU(2)$ YM theory in this background. Section III is devoted to obtaining the formal solutions of the self-duality equations for the general case, whereas subsections III A and III B deal with the form that these solutions take when the Weyl and the Lorenz gauge conditions, respectively, are employed. The expressions for the topological charge and the potential are given in section IV. In section V we start by discussing how the vacuum is constructed in the Euclidean AdS space. We construct meron-like solutions and examine their physical properties in subsection V A. We next study the continuous charge solutions numerically and explain their general features in subsection V B. Finally we conclude with section VI.

II.   SELF-DUALITY ON EUCLIDEAN SPHERICALLY SYMMETRIC SPACES

We consider static, spherically symmetric Euclidean spaces in Schwarzschild coordinates:

$$ds^2 = H(r)dt^2 + \frac{dr^2}{H(r)} + r^2 \left(d\theta^2 + \sin^2\theta\,d\phi^2\right).$$

---

[1] See [15] for the three dimensional version of this problem.

We take the $SU(2)$ Yang-Mills theory with the standard spherically symmetric instanton ansatz for the gauge connection [21]

$$A = \frac{\tau^a}{2}\left(\frac{x_a}{r}A_0\, dt + \frac{x_a x_i}{r^2}A_1\, dx^i + \frac{\phi_1}{r}\left(\delta_{ai} - \frac{x_a x_i}{r^2}\right)dx^i + \epsilon_{aij}\frac{\phi_2 - 1}{r^2}x^i dx^j\right),\quad (1)$$

where τ^a are the Pauli matrices. The four functions A_0, A_1, ϕ_1 and ϕ_2 depend on t, r only. It is important to note that a choice of gauge at this stage (such as $x^j A_j^a = A_1 = 0$) is not very convenient since this might lead to ostensibly time-dependent solutions, even though they yield a constant YM potential [10].

The four dimensional YM action can be reduced to a two dimensional Abelian Higgs model in a curved background as

$$I = \int_M d^4x\, \mathrm{tr}\,(F \wedge *F) = 4\pi \int_\Sigma d^2x\, \sqrt{\gamma}\left(\gamma^{\mu\nu}D_\mu\phi_a\, D_\nu\phi_a + \frac{1}{4}\gamma^{\mu\alpha}\gamma^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} + \frac{1}{2}(1 - \phi_a^2)^2\right),\quad (2)$$

where spacetime indices μ, ν refer to (t, r) and a, b run over $1, 2$; $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu\phi_a = \partial_\mu\phi_a + \epsilon_{ab}A_\mu\phi_b$ denote the two dimensional Abelian field strength and the covariant derivative, respectively. Here Σ stands for some suitable region, depending on $H(r)$, in the upper half plane with the metric:

$$ds^2 = \gamma_{\mu\nu}dx^\mu dx^\nu = \frac{H(r)}{r^2}dt^2 + \frac{dr^2}{r^2 H(r)}\,.$$

This type of reduction is of course well-known. [See [22] and the references therein.] One can work with the Abelian Higgs model without any loss of generality as guaranteed by Palais' symmetric criticality [23]. Either directly from the Abelian Higgs model or from the original four dimensional YM theory, the (anti) self-duality equations $F = \epsilon * F$ lead to

$$\dot{A}_1 - A_0' = -\epsilon\frac{1}{r^2}(1 - \phi_1^2 - \phi_2^2)\,,\quad (3)$$

$$\dot{\phi}_2 - A_0\,\phi_1 = \epsilon H(r)(\phi_1' + A_1\,\phi_2)\,,\quad (4)$$

$$\dot{\phi}_1 + A_0\,\phi_2 = -\epsilon H(r)(\phi_2' - A_1\,\phi_1)\,,\quad (5)$$

where $\epsilon = +1$ yields the self-dual and $\epsilon = -1$ the anti self-dual choice. Here we have denoted derivatives with respect to t and r with an overdot and a prime, respectively.

Before we move on the solutions of these equations, let us note that the left-over $U(1)$ symmetry of this model comes from the $SU(2)$ gauge transformations of the specific form $U(\hat{x}, t) = \exp\left(-i(\lambda(r, t)/2)\,\hat{x}\cdot\vec{\tau}\right)$. The effect of this gauge transformation on A_μ and ϕ_a is clear. As a side remark, let us also note that once this left-over symmetry is employed in eliminating one of these four functions, the remaining three, which depend on (t, r), define a surface in a three dimensional space. A close scrutiny shows that for the flat space case of $H(r) = 1$, the (anti) self-dual equations (3)–(5) are equivalent to the equations describing minimal surfaces in every aspect, including the topological charge [24, 25]. For $H(r) \neq 1$, the corresponding surfaces are not minimal in the (t, r) coordinates [16].

III.   THE SOLUTION WITHOUT A GAUGE CHOICE

One can treat (4) and (5) as a linear system of equations for A_0 and A_1 to quickly solve for these as

$$A_1 = \epsilon\frac{\dot{f}}{H(r)f} - \frac{g'}{1 + g^2}\quad \text{and}\quad A_0 = -\frac{\dot{g}}{1 + g^2} - \epsilon H(r)\frac{f'}{f}\,,$$

by defining $f^2 \equiv \phi_1^2 + \phi_2^2$ and $g \equiv \phi_1/\phi_2$. Using these in (3) then yields

$$\ddot{\omega} + H(r)(H(r)\omega')' = \frac{H(r)}{r^2}(e^{2\omega} - 1),\tag{6}$$

where we have also defined $\omega \equiv \ln f = \ln\sqrt{\phi_1^2 + \phi_2^2}$. The form of (6) hints at the Liouville equation which also shows up in the flat space choice $H(r) = 1$ [21]. In what follows, we will solve (6) by making some redefinitions and introducing new variables.

Now let $\omega = N(t, r) + h(r)$, where $h(r)$ is to be chosen. Then (6) becomes

$$\ddot{N} + H(r)(H(r)N')' = \frac{H(r)}{r^2}e^{2(N+h)} - H(r)\left(\frac{1}{r^2} + (H(r)h')'\right).\tag{7}$$

So given $H(r)$, one can choose $h(r)$ such that $H(r)h' = c + 1/r$, for some integration constant c, to get rid of the last term in (7). Moreover, if one introduces a new variable $\rho = \rho(r)$ such that $d\rho/dr = 1/H(r)$, then $\partial N/\partial\rho = H(r)N'$ and $\partial^2 N/\partial\rho^2 = H(r)(H(r)N')'$ in general. Thus, employing such a $\rho(r)$, the left hand side of (7) becomes $\partial^2 N/\partial t^2 + \partial^2 N/\partial\rho^2$.

Having Euclidean (A)dS space in mind, let us now introduce $\kappa = \pm 1$ (independent of ϵ) and take $H(r) = 1 - \kappa r^2/\ell^2$. Following the steps outlined above, one then finds

$$h(r) = \begin{cases} c\ell\tanh^{-1}(r/\ell) + \ln(r/\sqrt{r^2 - \ell^2}) + k, & \kappa = +1 \\ c\ell\tan^{-1}(r/\ell) + \ln(r/\sqrt{r^2 + \ell^2}) + k, & \kappa = -1 \end{cases}, \quad \text{with a new integration constant } k.$$

We can choose the constants c and k by keeping in mind that as $\ell \to \infty$, $h(r) \to \ln r$ to recover the flat space result. This forces us to set $c = 0$ and $k = \ln\ell$. [Obviously this argument is valid for the AdS case. One has to work with purely imaginary k in the dS case.] Moreover, one now has

$$\rho(r) = \begin{cases} \ell\tanh^{-1}(r/\ell), & \kappa = +1 \\ \ell\tan^{-1}(r/\ell), & \kappa = -1 \end{cases},\tag{8}$$

transforming (7) to the celebrated Liouville equation

$$\frac{\partial^2 N}{\partial t^2} + \frac{\partial^2 N}{\partial\rho^2} = -\frac{\kappa}{\ell^2}e^{2N(t,\rho)},\tag{9}$$

whose most general solution is

$$N(t,\rho) = \ln\left(\frac{2\,\ell\,|d\Phi(z)/dz|}{1 + \kappa|\Phi(z)|^2}\right),\tag{10}$$

where $\Phi(z)$ is an arbitrary analytic function of its complex argument $z = \rho(r) + it$ such that $d\Phi(z)/dz \neq 0$. Using these, one thus obtains

$$f^2(t,r) = \phi_1^2 + \phi_2^2 = \frac{4\,\ell^2\,r^2\,|d\Phi(z)/dz|^2}{(r^2 - \kappa\ell^2)(1 + \kappa|\Phi(z)|^2)^2}.\tag{11}$$

A. The Weyl Gauge

So far, we have not used the gauge invariance of the action (or the field equations). [See the paragraph containing equation (11) of [10] for details.] Fixing the gauge, we can find the unknown functions. For example, employing the Weyl gauge, $A_0 = 0$, one gets the following equation for the unknown g:

$$\frac{\partial(\tan^{-1}(g))}{\partial t} = -\epsilon H(r)\frac{\partial(\ln f(t,r))}{\partial r},$$

which can be solved explicitly given $\Phi(z)$. Using the solution $g(t, r)$, A_1 is found as

$$A_1 = \frac{\epsilon}{H(r)} \frac{\partial(\ln f(t, r))}{\partial t} - \frac{\partial(\tan^{-1}(g))}{\partial r} \, .$$

Likewise, one obtains

$$\phi_1 = \frac{fg}{\sqrt{1 + g^2}} \quad \text{and} \quad \phi_2 = \frac{f}{\sqrt{1 + g^2}} \, .$$

In the Hamiltonian processes, such as tunneling, the Weyl gauge is quite useful. However, in what follows, we will mainly work in the Lorenz gauge $\partial_\mu(\sqrt{\gamma} A^\mu) = 0$, which is somewhat more convenient in finding the solutions.

B. The Lorenz Gauge

From now on we will employ the Lorenz gauge and concentrate only on $\kappa = -1$, i.e. the case of AdS space. Using $\sqrt{\gamma} = 1/r^2$, the Lorenz gauge condition can be solved easily as $A^0 = -\epsilon r^2 \chi'$ and $A^1 = \epsilon r^2 \dot\chi$ for some function $\chi(t, r)$. Defining ψ_a as $\phi_a = e^\chi \psi_a$ now reduces the system (3)–(5) to

$$\frac{\ddot\chi}{H(r)} + (H(r)\chi')' = \frac{1}{r^2}\left(e^{2\chi}(\psi_1^2 + \psi_2^2) - 1\right), \tag{12}$$

$$\dot\psi_2 = \epsilon H(r)\psi_1', \tag{13}$$

$$\dot\psi_1 = -\epsilon H(r)\psi_2', \tag{14}$$

respectively. If one further introduces a new variable $\rho = \rho(r)$ as before, such that $d\rho/dr = 1/H(r)$, and $\psi(z) = \psi_1 + i\psi_2$, where $z = \rho + i\epsilon t$, then (13) and (14) can be thought of as the Cauchy-Riemann conditions that $\psi(z)$ has to satisfy to be analytic. Now when $H(r) = 1 + r^2/\ell^2$, $\rho(r)$ is given by (8). The remaining equation (12) becomes

$$\frac{\partial^2 \chi}{\partial t^2} + \frac{\partial^2 \chi}{\partial \rho^2} = (\Omega(\rho))^2\left(e^{2\chi(t,\rho)}|\psi|^2 - 1\right), \quad \text{where} \quad 1/\Omega(\rho) \equiv \ell \sin(\rho/\ell) \, .$$

Note that $(\partial_t^2 + \partial_\rho^2)\ln|\psi|^2 = 0$ for any analytic function ψ of $z = \rho + i\epsilon t$, except at isolated singularities. Moreover, $\partial_\rho^2 \ln(\ell\Omega(\rho)) = (\Omega(\rho))^2$. Using this freedom, we can set

$$\chi(t, \rho) = -\frac{1}{2}\ln|\psi|^2 - \ln(\ell\Omega(\rho)) + N(t, \rho)$$

to arrive at the Liouville equation (9) (recall that we have $\kappa = -1$) with the generic solution (10). Finally one has

$$\chi(t, \rho) = \ln\left(\frac{2\,|d\Phi(z)/dz|}{(1 - |\Phi(z)|^2)\,\Omega(\rho)\,|\psi|}\right) \, .$$

Now the question is how to choose the analytic function $\psi(z)$. Guided by the flat space analysis of [21], we set $\psi(z) = d\Phi(z)/dz$ for now. One then finds

$$\phi_1^2 + \phi_2^2 = \frac{4\,|d\Phi(z)/dz|^2}{(\Omega(\rho))^2(1 - |\Phi(z)|^2)^2} \, , \tag{15}$$

which is consistent with (11).

IV. THE CHARGE AND THE POTENTIAL

Before we move on the construction of the explicit solutions, let us write down the charge and the potential in terms of the reduced fields, taking into account the boundary terms. These will be necessary for the discussion of the physical properties of the solutions.

Defining $\epsilon_{tr} = 1/r^2$ and $\epsilon_{12} = 1$, the field equations coming from the variation of the 2-dimensional action (2) are

$$D_\mu \phi_a = -\gamma_{\mu\alpha} \, \epsilon^{\alpha\nu} \, \epsilon_{ab} D_\nu \phi_b \,, \tag{16}$$
$$F_{\mu\nu} = -\epsilon_{\mu\nu}(1 - \phi_a^2) \,, \tag{17}$$

which are identical to the set (3)–(5) for the choice $\epsilon = +1$. Using these in the action (2) and taking the boundary term

$$-4\pi \int_\Sigma d^2x \partial_\alpha \left(\sqrt{\gamma} \epsilon^{\nu\alpha} \epsilon_{ab} \phi_b D_\nu \phi_a \right) \to 0$$

at $r \to \infty$, one finds

$$I = 4\pi \int_\Sigma d^2x \sqrt{\gamma}(1 - \phi_a^2) \,.$$

The topological charge is thus given as

$$Q = \frac{1}{8\pi^2} \int_M d^4x \, \mathrm{tr}\,(F \wedge F) = \frac{I}{8\pi^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_0^{\infty} dr \, \frac{1}{r^2}(1 - \phi_1^2 - \phi_2^2) \,. \tag{18}$$

To really appreciate the physics of the solutions obtained, one also needs the gauge invariant YM potential which reads [6]

$$V(t) = 2\pi \int_0^{\infty} dr \left(2H(r)(\phi_1' + A_1\,\phi_2)^2 + 2H(r)(\phi_2' - A_1\,\phi_1)^2 + \frac{1}{r^2}(1 - \phi_1^2 - \phi_2^2)^2 \right) \,. \tag{19}$$

V. THE SOLUTIONS

We have seen in section III B that, given an analytic function $\Phi(z)$, one can construct a gauge field A (1) which is (anti) self-dual. However, not all (anti) self-dual solutions will have finite action. For example, following [21], consider the meromorphic function that leads to the vacuum in flat space

$$\Phi(z) = \frac{a - z}{\bar{a} + z} \,, \quad \text{where} \quad a \in \mathbb{C} \quad \text{and for which} \quad \frac{d\Phi}{dz} = -\frac{2\,\Re(a)}{(\bar{a} + z)^2} \,.$$

This choice of $\Phi(z)$ also gives a self-dual solution in Euclidean AdS space. In fact, $\Phi(z)$ above yields

$$f^2(t, r) = \frac{r^2}{(r^2 + \ell^2)(\tan^{-1}(r/\ell))^2}$$

and

$$Q = \int_{-\infty}^{\infty} dt \int_0^{\infty} dr \, \frac{1}{r^2}(1 - f^2) = \int_{-\infty}^{\infty} dt \, \frac{2}{\ell\pi} \,,$$

which is clearly divergent. Hence, not all $\Phi(z)$ is allowed. We have to consider only those analytic functions that lead to finite action (or charge) solutions.

In search of these analytic functions, the representation of the vacuum plays a crucial role. In flat space, once the vacuum is properly represented, multi-instanton solutions can be obtained simply by taking the suitable products of the $\Phi(z)$ that corresponds to it, i.e. they are obtained from $\prod_{i=1}^{k} \frac{a_i-z}{\bar{a}_i+z}$. Just as in the flat space case, the vacuum in our setting is clearly given by $\phi_1^2 + \phi_2^2 = 1$, for which (18) yields $Q = 0$ for the charge. Clearly $\phi_2 = 1$ and $\phi_1 = A_0 = A_1 = 0$ is the trivial vacuum $A = 0$ (1). However, finding the analytic function $\Phi_v(z)$ (where the subscript 'v' refers to the vacuum) that gives $A = 0$ is somewhat non-trivial. From (15), we find that the function $\Phi_v(z)$ can be chosen as

$$\Phi_v(z) = -\tan\left(\frac{z}{2\ell} - \frac{\pi}{4}\right) = \frac{1 - \tan\left(z/2\ell\right)}{1 + \tan\left(z/2\ell\right)}, \tag{20}$$

which is analytic everywhere except at $z = 2\ell\pi(k + 3/4)$ for $k \in \mathbb{Z}$. Moreover, one has

$$\lim_{\ell \to \infty} \Phi_v(z) = 1, \quad \text{and} \quad |\Phi_v|^2 = \frac{\sin^2(-\pi/4 + \rho/2\ell) + \sinh^2(t/2\ell)}{\cos^2(-\pi/4 + \rho/2\ell) + \sinh^2(t/2\ell)}.$$

In fact, one can dress up this function with a complex parameter a, thanks to the invariance of the solutions (15) under the Möbius transformation

$$\Phi(z) \to \frac{c + \Phi(z)}{\bar{c}\Phi(z) + 1} \quad \text{for any} \ \ c \in \mathbb{C},$$

to get

$$\tilde{\Phi}_v(z) = \frac{a - \tan\left(z/2\ell\right)}{\bar{a} + \tan\left(z/2\ell\right)} \quad \text{with} \quad a \in \mathbb{C}. \tag{21}$$

The latter yields

$$\phi_1 + i\phi_2 = e^{\chi} \frac{d\tilde{\Phi}_v}{dz} = -\frac{\bar{F}}{F},$$

where $F(z) = \bar{a}\cos\left(z/2\ell\right) + \sin\left(z/2\ell\right)$. Let us now show that $\tilde{\Phi}_v(z)$ is the function that leads to the trivial vacuum $A = 0$. Recalling that even within the Lorenz gauge, one still has the freedom of choosing $\psi(z)$ in the solution for $\chi(t, \rho)$, we set $\psi(z) = w(z)\, d\tilde{\Phi}_v(z)/dz$, where $w(z)$ is an analytic function of z, leading to

$$\phi_1 + i\phi_2 \to (\phi_1 + i\phi_2)\frac{w(z)}{|w(z)|}.$$

Setting $w(z) = -iF^2$ clearly gets one to the vacuum $\phi_1 + i\phi_2 = i$. Thus (21) gives the vacuum in the Euclidean AdS space.

As in the flat space, we will construct the finite action solutions using (21). However, in contrast to flat space, here the action depends on the parameters a_i in a non-trivial way. This is to be expected since we are in AdS space with an intrinsic length scale. [Recall that in flat space, the parameters a_i determine the size and the locations of the instantons [21]. In AdS, the existence of the boundary at a finite distance (as explained in the penultimate paragraph of section I) drastically modifies the dependence of the action on the instanton moduli.] Hence the following function

$$\prod_{i=1}^{k} \frac{a_i - \tan\left(z/2\ell\right)}{\bar{a}_i + \tan\left(z/2\ell\right)}$$

leads to a finite action self-dual solution. The topological charge and the action depend on the a_i in a non-trivial way and, unfortunately, the action can only be calculated numerically for generic a_i.

A. The meron-like solutions

In the special case of $a_i = 1$, the calculations can be carried out analytically. For example, consider $\Phi(z) = (\Phi_v(z))^2$. Then it is not hard to show that (15) yields

$$\phi_1^2 + \phi_2^2 = \frac{4|\Phi_v|^2}{(1 + |\Phi_v|^2)^2}, \quad \text{and} \quad 1 - \phi_1^2 - \phi_2^2 = \left(\frac{\cos\left(\rho/\ell - \pi/2\right)}{\cosh\left(t/\ell\right)}\right)^2.$$

Using this in (18) gives $Q = 1/2$ since

$$\int_{-\infty}^{\infty} dt \int_0^{\infty} dr \, \frac{1}{r^2}(1 - \phi_1^2 - \phi_2^2) = \pi.$$

Similarly, one can also work out the potential in this case. From (19) it simply reads

$$V(t) = \frac{3\pi^2}{2\ell} \operatorname{sech}^4\left(t/\ell\right).$$

Other examples of half-integer charges can be constructed in this vein, however calculations get rather complicated. We were able to show that if one chooses $\Phi(z) = (\Phi_v(z))^n$, where $n = 2^k$ for $k \in \mathbb{Z}^+$, then $Q = (n-1)/2$. Note that these are genuinely new and non-trivial solutions in the Euclidean AdS space and completely disappear in the flat space limit $\ell \to \infty$.

In flat space, charge-1/2 solutions of the *full* YM equations exist and go under the name as 'merons' [26]. Note however that these are singular solutions with a divergent action. Additionally, note also that charge-3/2 *self-dual* solutions in flat space were constructed as well [17]. Here, we have shown that the Euclidean AdS space admits similar half-integer meron-like solutions with a finite action.

B. The continuous charge solutions

Let us now consider more general solutions. Let

$$\Phi(z) = (\tilde{\Phi}_v(z))^2 = \left(\frac{a - \tan\left(z/2\ell\right)}{\bar{a} + \tan\left(z/2\ell\right)}\right)^2$$

with $a \equiv \alpha + i\beta$, a complex parameter. Then the relevant integrand for the action, charge or the potential energy reads

$$\frac{1}{r^2}(1 - \phi_1^2 - \phi_2^2) = \frac{16\alpha^2 \sin^2\left(\rho/\ell\right)|\cos\left(z/2\ell\right)|^4}{r^2\left(4\alpha^2|\cos\left(z/2\ell\right)|^4 + \sin^2\left(\rho/\ell\right) + \left(\sinh\left(t/\ell\right) - 2\beta|\cos\left(z/2\ell\right)|^2\right)^2\right)^2}, \quad (22)$$

where $|\cos\left(z/2\ell\right)|^2 = \cosh^2\left(t/2\ell\right) - \sin^2\left(\rho/2\ell\right)$. Unfortunately, the computations from this point on can only be performed numerically. In figures 1 and 2, we have calculated the topological charge Q and the potential $V(t)$, respectively, to exhibit the general features of the solutions obtained using (22).

Fig. 1 depicts the topological charge Q as a function of α and β. When $\alpha = 0$, $Q = 0$ as expected since α (as discussed below) is a parameter that gives the scale of the solution, and when

$\alpha = 0$, the solution becomes trivial. Even though it is not very apparent in Fig. 1, the charge Q changes very slowly with β. Note that as long as $\ell \neq 0$ or ∞, Q does not depend on ℓ. [Note that the discontinuity at $\alpha = 0$ is natural, since $\alpha \to 0$ limit is formally equivalent to the $\ell \to \infty$ limit for which we have $Q \to 1$, which is the flat space result. However, when α is exactly zero to start with, a careful analysis gives $Q = 0$ as explained above.]
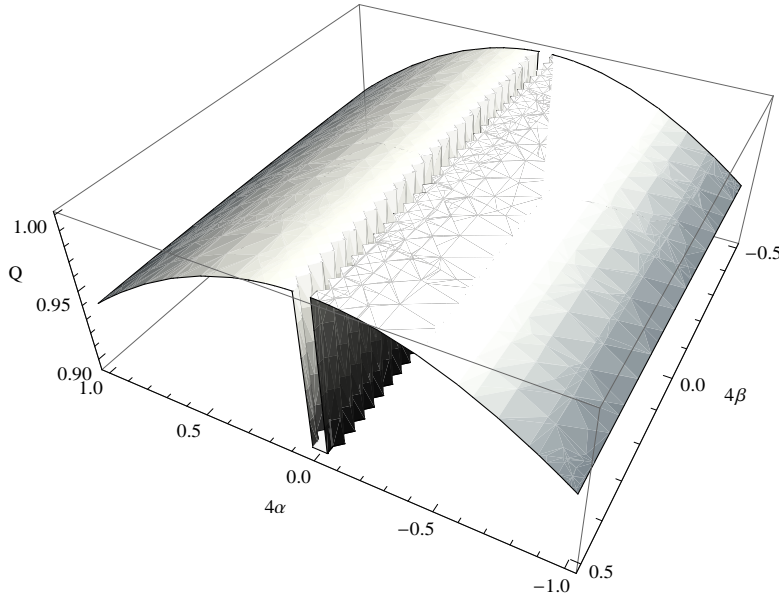

FIG. 1: The topological charge Q when $\ell = 2$.

In Fig. 2, we have plotted the potential $V(t)$ as a function of time t for fixed $\ell = 2$. As argued below, β determines the 'location' of the solution on the time t-axis.

It is worth emphasizing that in the flat space limit $\ell \to \infty$, one obtains the corresponding flat space solution after redefining $a \to 2\ell a$. [That is why we have 4α and 4β in the labels of the axes in Fig. 1.] The interpretation of the parameters a_i, in terms of the 'scale' and the location (on the t-axis), follows the discussion in flat space. Because of our choice, a_i are dimensionless of course, but ℓ clearly acts as the proper length parameter. Let us define the 'location' of the solution as the point on the time t-axis where the potential energy takes its maximum value . [For multi-instantons, the maxima of the potential define the individual locations of the 'instantons'.] From (22), it follows that for such maxima one should look for the solutions of $\tilde{\Phi}_v(z) \frac{d\tilde{\Phi}_v(z)}{dz} = 0$. One can check that this boils down to finding the zeros of $\tilde{\Phi}_v(z) = 0$, since its derivative does not vanish in the relevant domain. Hence, one should solve $\alpha + i\beta = \tan((\rho_0 + it_0)/2\ell)$, which yields

$$\alpha = \frac{\tan(\rho_0/2\ell) \, \text{sech}^2(t_0/2\ell)}{1 + \tan^2(\rho_0/2\ell) \, \tanh^2(t_0/2\ell)}, \qquad \beta = \frac{\tanh(t_0/2\ell) \, \sec^2(\rho_0/2\ell)}{1 + \tan^2(\rho_0/2\ell) \, \tanh^2(t_0/2\ell)}. \qquad (23)$$

Given α and β, one immediately finds the scale ρ_0 and the location t_0. It is important to note that unlike the flat space case, here α and β are restricted to the domains $\alpha \in [0,1]$ and $\beta \in [-1,1]$. This follows from (23) by a careful consideration of the ranges of $\rho \in [0, \pi\ell/2)$ and $t \in (-\infty, \infty)$. Specifically, consider $\Phi(z) = (\Phi_v(z))^2$, i.e. $\alpha = 1$ and $\beta = 0$ case, studied in subsection V A. This corresponds to the case of having $t_0 = 0$ and the size of the 'meron' going to infinity, that is the 'meron' fills the whole space.
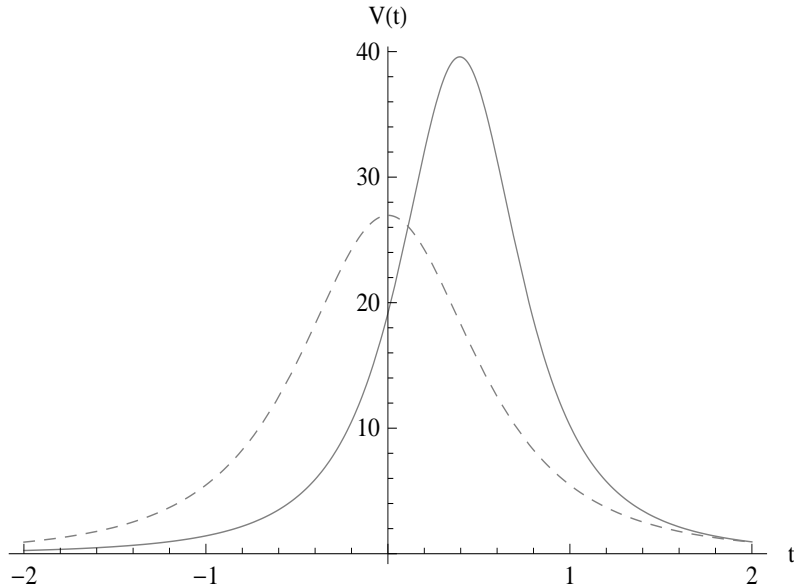
FIG. 2: The potential $V(t)$ as a function of t when $\ell = 2$. The solid line is for $\alpha = 1/8$, $\beta = 1/10$; whereas the dashed line is for $\alpha = 3/16$ and $\beta = 0$.

To keep the discussion simple, we have refrained from considering either

$$\Phi(z) = \prod_{i=1}^{k} \frac{a_i - \tan(z/2\ell)}{\bar{a}_i + \tan(z/2\ell)} ,$$

or higher powers of $\tilde{\Phi}_v(z)$, i.e. $\Phi(z) = (\tilde{\Phi}_v(z))^k$ with $k \geq 3$, here. Except for the $a = 1$ case, which is studied in subsection V A, we have not been able to compute either the charge or the potential analytically. However, it is clear by construction that these also lead to (anti) self-dual solutions and in principle it is possible to numerically obtain the physical properties of these as well.

VI. CONCLUSIONS

We have studied the (anti) self-dual $SU(2)$ gauge fields in the Euclidean Anti-de Sitter space. We have shown that the problem eventually reduces to finding the solutions of the Liouville equation on the strip $0 \leq \Re(z) < \pi\ell/2$ of the complex plane. We have seen that given *any* analytic function, one can construct (anti) self-dual solutions which do not in general have finite action. Finding finite action (or charge) solutions reduces to finding *proper* analytic functions inside the relevant strip as discussed in section V. The solutions we have found have quite interesting properties: They can have any non-integer charge including fractional values. Our solutions depend on the time coordinate t and have non-trivial YM potential. In this respect, they are quite distinct than the earlier, *static* solutions [16]. Looking at the potential, one can see that the solutions presented here resemble pretty much the flat space instantons, having $V(t \to \pm\infty) = 0$ and a bump (or bumps in between). We have also explained how a non-integer charge is quite natural in the AdS context. [Note that even in flat space, non-integer charge values are allowed [17, 18].]

In the search of the solutions, we have left one question unanswered: Are there generic integer-charge solutions? There seems to be no compelling reason why there should not be any. Unfortunately though, we have not been able to find these solutions. It is quite interesting that certain

fractionally charged solutions appear more naturally in AdS than the integer ones. A further direction of research would be to consider the Euclidean de Sitter (dS) space. It is clear that most of the equations in this paper also work for the dS space. The problem arises again in finding the proper analytic functions that will yield finite action solutions. In dS, because of the cosmological horizon, one has to search for time-periodic solutions, namely finite temperature caloron solutions, restricted to live inside the horizon.

VII.   ACKNOWLEDGMENTS

[1] M.A. Shifman, "Instantons in gauge theories," *Singapore, Singapore: World Scientific (1994) 488 p.*
[2] G.'t Hooft, Commun. Math. Phys. **81**, 267 (1981).
[3] S.A. Cherkis, "Instantons on the Taub-NUT Space," arXiv:0902.4724 [hep-th].
[4] D. Harland, Commun. Math. Phys. **280**, 727 (2008) [arXiv:hep-th/0703277].
[5] B.J. Harrington and H.K. Shepard, Phys. Rev. D **17**, 2122 (1978).
[6] G.V. Dunne and B. Tekin, Phys. Rev. D **63**, 085004 (2001) [arXiv:hep-th/0011169].
[7] K. Lee and C. Lu, Phys. Rev. D **58**, 025011 (1998) [arXiv:hep-th/9802108].
[8] T.C. Kraan and P. van Baal, Nucl. Phys. B **533**, 627 (1998) [arXiv:hep-th/9805168].
[9] M.F. Atiyah, N.J. Hitchin and I.M. Singer, Proc. Roy. Soc. Lond. A **362**, 425 (1978).
[10] B. Tekin, Phys. Rev. D **65**, 084035 (2002) [arXiv:hep-th/0201050].
[11] J.M. Charap and M.J. Duff, Phys. Lett. B **69**, 445 (1977).
[12] J.M. Charap and M.J. Duff, Phys. Lett. B **71**, 219 (1977).
[13] E. Radu, D.H. Tchrakian and Y. Yang, Phys. Rev. D **77**, 044017 (2008) [arXiv:0707.1270 [hep-th]].
[14] G. Etesi and M. Jardim, Commun. Math. Phys. **280**, 285 (2008) [arXiv:math/0608597].
[15] A. Ferstl, B. Tekin and V. Weir, Phys. Rev. D **62**, 064003 (2000) [arXiv:hep-th/0002019].
[16] H. Boutaleb-Joutei, A. Chakrabarti and A.Comtet, Phys. Rev. D **20**, 1884 (1979).
[17] P. Forgacs, Z. Horvath and L. Palla, Phys. Rev. Lett. **46**, 392 (1981).
[18] E. Farhi, V.V. Khoze and R. Singleton, Phys. Rev. D **47**, 5551 (1993) [arXiv:hep-ph/9212239].
[19] Y. Verbin, Class. Quant. Grav. **7**, L89 (1990).
[20] J.M. Maldacena and L. Maoz, JHEP **0402**, 053 (2004) [arXiv:hep-th/0401024].
[21] E. Witten, Phys. Rev. Lett. **38**, 121 (1977).
[22] A.M. Jaffe and C.H. Taubes, "Vortices And Monopoles. Structure Of Static Gauge Theories," *Boston, USA: Birkhaeuser (1980) 287 p. (Progress In Physics, 2).*
[23] R.S. Palais, Commun. Math. Phys. **69**, 19 (1979).
[24] A. Comtet, Phys. Rev. D **18**, 3890 (1978).
[25] B. Tekin, JHEP **0008**, 049 (2000) [arXiv:hep-th/0006135].
[26] V.de Alfaro, S. Fubini and G. Furlan, Phys. Lett. B **65**, 163 (1976).